Spectral Geometry Processing with Manifold Harmonics

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Bruno Lévy
Introduction

Introduction

Ⅱ. Harmonics
Ⅲ. DEC formulation
Ⅳ. Filtering
Ⅴ. Numerics

Results and conclusion
Introduction

Extend to meshes:

- Fourier transform
- Spectral filtering
Introduction

Fourier transform

\[ f = \sum \sin(kx) \]
**Introduction**

**Filtering**

Fourier Transform

- Geometric space
- Frequency space

Convolution

- Filtering

Inverse Fourier Transform

- X

FP4 - 16/04/2008

Eurographics 2008
Filtering on a mesh

Introduction

Filtering
[Taubin 95]

Geometric space
Frequency space
Introduction

Filtering on a mesh

Filtering

[Taubin 95]

Geometric space

Frequency space

?  x  ?

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Eurographics 2008
Introduction

Filtering on a mesh

Filtering
[Taubin 95]

Geometric space
Frequency space

[Karni00] mesh compression
[Zhang06] shape matching
[Dong06] quadrangulation

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Eurographics 2008
I Harmonics

Introduction
- Harmonics
- DEC formulation
- Filtering
- Numerics

Results and conclusion
Question

I Harmonics

\[ \sin(kx) \]
Harmonics and vibrations

\[ \sin(kx) \] are the stationary vibrating modes = harmonics of a string
Manifold Harmonics

Harmonics

Harmonics

?
I Harmonics

Chladni plates

sand

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I Harmonics

Chladni plates
Discoveries concerning the theory of music, Chladni, 1787
Chladni plates and jpeg

Chladni plates, 1787

Discrete cosine transform (jpeg)
Spherical Harmonics

I Harmonics

Harmonics

Harmonics

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Manifold Harmonics

Stationary vibrating modes
Harmonics and vibrations

- **Wave equation:**
  \[ T \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \]
  \( T \): stiffness \( \mu \): mass

- **Stationary modes:**
  \[ y(x,t) = y(x)\sin(\omega t) \]
  \[ \frac{\partial^2 y}{\partial x^2} = -\mu\omega^2/T \quad y \]
  eigenfunctions of \( \frac{\partial^2}{\partial x^2} \)
I Harmonics

Harmonics: recap

- Harmonics are **eigenfunctions** of $\frac{\partial^2}{\partial x^2}$
- On a mesh, $\frac{\partial^2}{\partial x^2}$ is the Laplacian $\Delta$
- We need the **eigenfunctions** of $\Delta$
- Let’s use **DEC**
II DEC formulation

Introduction
- Harmonics
- DEC formulation
- Filtering
- Numerics

Results and conclusion
II DEC formulation

Discrete Exterior Calculus (DEC)

Discretize equations on a mesh
• Simple
• Rigorous

[Mercat], [Hirani], [Arnold], [Desbrun]

Based on k-forms
II DEC formulation

k-forms

mesh
dual mesh
II DEC formulation

0-forms

-7.5 -4.6 3.5 5.1
0.2 3.5 -2.7 5
3.5 5
1-forms

II DEC formulation
II DEC formulation

2-forms
II DEC formulation

dual 0-forms
II DEC formulation

Dual 1-forms
II DEC formulation

dual 2-forms
Hodge star $\star_0$

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>term</th>
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<tbody>
<tr>
<td>0-forms</td>
<td>dual 2-forms</td>
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II DEC formulation

mesh
dual mesh
### Hodge star $\star_1$

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<th>term</th>
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</thead>
<tbody>
<tr>
<td>1-forms</td>
<td>dual 1-forms</td>
<td>$</td>
</tr>
</tbody>
</table>

**Diagram:**
- Mesh
- Dual mesh
- $i$, $j$, $\beta$, $\beta'$
- Hodge star operator $\star_{ij}$
### Exterior derivative $d$

The exterior derivative is a fundamental concept in differential geometry. It is denoted by $d$ and maps $k$-forms to $(k+1)$-forms. The exterior derivative of a 1-form $\alpha = f_i dx^i$ is defined as $d\alpha = df = \sum_j \partial_j f_i \, dx^i \wedge dx^j$, where $\partial_j f_i$ are the components of the gradient of $f_i$.

#### II DEC formulation

The II DEC formulation is a compact way to represent the exterior derivative in terms of the oriented connectivity of the mesh. It is given by:

$$df (ij) = f_i - f_j$$

where $f_i$ are the function values at the vertices of the mesh.

### Table

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<tbody>
<tr>
<td>0-forms</td>
<td>1-forms</td>
<td>$df (ij) = f_i - f_j$</td>
</tr>
</tbody>
</table>

### Diagram

Oriented connectivity of the mesh:

- $f_i$: function at vertex $i$
- $f_j$: function at vertex $j$
- $f_k$: function at vertex $k$
- $f_l$: function at vertex $l$

The connectivity is represented by the table:

<table>
<thead>
<tr>
<th></th>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>$l$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ij$</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>$f_i$</td>
</tr>
<tr>
<td>$jk$</td>
<td>0</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
<td>$f_j$</td>
</tr>
<tr>
<td>$ki$</td>
<td>+1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$f_k$</td>
</tr>
<tr>
<td>$il$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>$f_l$</td>
</tr>
<tr>
<td>$lj$</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

This table illustrates how the exterior derivative is computed for each edge of the mesh, taking into account the orientation of the mesh.
In DEC the Laplacian is \( *_{0}^{-1} d^T *_{1} d \)

0-form (function) \( f \)
In DEC the Laplacian is $\star_0^{-1} d^T \star_1 d$

0-form (function) $f$

$d$

1-form (gradient) $df$

$(f_j - f_i)$
In DEC the Laplacian is

$$\star_0^{-1} d^T \star_1 d$$

0-form (function) $f$

1-form (gradient) $df$

$$(\cot(\beta) + \cot(\beta')) (f_j - f_i)$$

$d$

$\star_1$

dual 1-form (cogradient) $\star df$
In DEC the Laplacian is $\star_{0^{-1}} d^T \star_1 d$

0-form (function) $f$

$\sum (\cot(\beta)+\cot(\beta'))(f_j-f_i)$

dual 0-form (integrated laplacian) $d \star df$

dual 1-form (cogradient) $\star df$

1-form (gradient) $df$
In DEC the Laplacian is

\[ \star_0^{-1} d^T \star_1 d \]

0-form (function) \( f \)
0-form (pointwise laplacian)

\[ \star_0^{-1} d \star d f \]

\[ \sum_i (\cot(\beta) + \cot(\beta')) (f_j - f_i) \]

|\( *i \)|

1-form (gradient) \( df \)

dual 0-form (integrated laplacian) \( d \star df \)

dual 1-form (cogradient) \( \star df \)

\( \Sigma \)
Manifold Harmonics Basis (MHB)

Eigenfunctions of
operator $\Delta$

$\text{DEC}\downarrow$

Eigenvectors of
matrix $*_0^{-1}d^T*_1d$

$H^1$ $\ldots$ $H^{10}$ $\ldots$
$H^7$ $\ldots$ $H^{700}$ $\ldots$
$\ldots$ $H^{18}$ $\ldots$ $H^m$
II DEC formulation: recap

\[ \sin(kx) \] on \( H^1 \) to \( H^m \)
III Filtering

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Results and conclusion
III Filtering
III Filtering

Spectral Filtering

- The Manifold Harmonics $H^k$ come with an eigenvalue $\lambda_k$
- The $\lambda_k = \omega_k^2$ is a squared spatial frequency
- A filter is a transfer function $F(\omega)$

Geometric space

Frequency space

$$f_i \xrightarrow{\text{Filtering}} f_{i}^{\text{filt}}$$

$$f_k \xrightarrow{\text{MHT}} \beta_k \xrightarrow{F(\omega_k)} F(\omega_k) \beta_k$$

$$f^\sim_k \xrightarrow{\text{MHT}^{-1}} f_k$$
Take $f = (r, g, b)$
Take $f = (x, y, z)$
IV Numerics

Introduction

- Harmonics
- DEC formulation
- Filtering

Numerics

Results and conclusion
Eigenvalues

- Compute the eigenpairs \((H_k, \lambda_k)\) of \(L = *_0^{-1}d^T *_1 d\)
- Solver returns eigenvectors of highest eigenvalue

Problem:
- We want smallest \(\lambda_k\)
- We want more than 50

Compute \(x = L v\)

\((H_k, \lambda_k)\) of \(L\) for 50 largest \(\lambda_k\)
Shift Invert

- $L \rightarrow (L - \lambda_s \text{Id})^{-1}$
- same $H_k$

$$\mu_k = \frac{1}{\lambda_k - \lambda_s}$$
IV Numerics

Eigen solver

Compute a band of eigenpairs \((H^k, \lambda^k)\) around \(\lambda_s\)

\[
x = (L - \lambda_s \text{Id})^{-1} v
\]

Solve \(Lx - \lambda_s x = v\)

Eigen Solver

\((H_k, \mu_k)\) for 50 largest \(\mu_k\)

Solve indefinite Cholesky factorization + backward substitution

\([\text{Meshar} \& \text{Toledo}]\)

\((H_k, \lambda_k)\) for 50 \(\lambda_k\) closest to \(\lambda_s\)
Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band
IV Numerics

Band by band algorithm

Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band

\[ \lambda_s^0 = 0 \]
IV Numerics

Band by band algorithm

Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band

\[
\lambda_s^0 = 0 \quad \lambda_s^1
\]
IV Numerics

Band by band algorithm

Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band

\[\lambda_s^0 = 0 \quad \lambda_s^1 \quad \lambda_s^2\]
Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band
Band by band algorithm

Compute the eigenpairs \((h^k, \lambda^k)\) of \(L\) band by band.
Results and conclusion

Introduction

II. Harmonics

III. DEC formulation

IV. Filtering

V. Numerics

Results and conclusion
Results
Conclusion

We make explicit Fourier Analysis and Filtering tractable

Time to compute MHB ~ Time to compute a filter
(5 minutes for 300k vertices)

Time to update filter ~ real time
Acknowledgements

• Ramsay Dyer for personal communication

• Sivan Toledo for the sparse indefinite Cholesky factorization code
Questions ?